On the Hermitian momentum of Wigner-Dunkl quantum mechanics

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In this paper the Hermitian momentum operator on the usual Hilbert space is constructed for the Wigner-Dunkl quantum mechanics utilizing a symmetric Dunkl derivative. The inverse of the derivative is shown to exhibit different realization on the subspaces of even and odd functions. The continuity conditions at finite discontinuities of symmetric potential is investigated. As an example, the finite symmetric square well is discussed in detail.

I. INTRODUCTION

The foundation of quantum mechanics is still an attractive field of contemporary physics. Most notable, the 2022 physics Nobel prize was awarded for experiments with entangled photons [1]. For some recent theoretical work let us mention [2–4].

Another field of increasing interest is related to deformations of standard textbook quantum mechanics. One of such deformations, which is the subject of the present work, goes back to Wigner and Dunkl, and nowadays attracts much attention among the physics community. Being a bit more explicit, in 1950 Wigner [5] investigated the effects of the reflection operator on the spectrum of the harmonic oscillator. Shortly after, Yang [6] introduced a modified Heisenberg algebra by explicitly introducing the reflection operators with a deformation parameter resulting in a modified momentum operator. Independently of these studies, when investigating polynomials with discrete symmetry groups, Dunkl [7] obtained similar modified momentum operators. More recently such deformations of the momentum operator or Heisenberg algebra has attracted much attention in quantum mechanical system, often called Wigner-Dunkl quantum mechanics (WDQM). For example, in refs. [8–10] the Dunkl oscillator was investigated in two and three dimensions. The 2-dimensional Dunkl oscillator was also studied via the $su(1,1)$ algebraic approach in [11]. The 3-dimensional non-relativistic Dunkl -Coulomb problem and its superintegrability were investigated in ref. [12, 13]. In a recent work [14] the current authors revisited the Wigner-Dunkl quantum mechanics from a supersymmetric point of view, introduced a generalized shape invariance and provided exact solutions for a certain class of SUSY potentials. Dunkl derivatives with two and three parameters are investigated in refs. [15–17].

The momentum operator in WDQM has the problem of being not Hermitian on the usual Hilbert space $L^2(\mathbb{R}, \mathrm{d}x)$ equipped with the standard Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$. Hence, one usually introduces a weighted measure $\mathrm{d}\mu(x)$ curing this defect. The objective of the current work is two-fold. First we introduce a modified momentum operator for WDQM which is Hermitian on the standard Hilbert space $L^2(\mathbb{R}, \mathrm{d}x)$. Secondly, we want to study the spectral properties of WDQM for the finite potential well, which is one of the basic textbook problems not explicitly but graphically or numerically solvable. In doing so we need to investigate the continuity conditions of wave functions in WDQM at locations where the potential has a finite discontinuity.

This paper is organized as follows. In section 2 we recall some basic properties of WDQM paving the way for the construction of a modified but Hermitian momentum operator, which is explicitly discussed in section 3 in details. This section also touches upon the modified continuity equation for WDQM. In section 4 we deal with our second objective, that is, we explicitly study the continuity conditions of energy eigenfunction at points where the external potential exhibits a finite discontinuity. As an explicit example we then solve the WDQM eigenvalue problem for a finite potential well in section 5, and conclude in section 6 with some final

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remarks summarizing our findings.

II. PRELIMINARIES ON DUNKL QUANTUM MECHANICS

As pointed out above, the first deformation of Heisenberg's algebra involving the reflection operator is due to Wigner [5] and Yang [6]. What they proposed was a deformed commutation relation between position operator $\hat{x}$ and linear momentum operator $\hat{p}$ of the form

$$[\hat{x}, \hat{p}] = \mathrm{i}\,(1 + 2\nu R)\ . \tag{1}$$

In the above ν is an arbitrary real parameter with $\nu > -\frac{1}{2}$. R represents the reflection (or parity) operator acting on functions f defined on the real line as $(Rf)(x) = f(-x)$ and thus is a non-local operator. Obviously, the parity operator anti-commutes with both, the position operator and the linear momentum operator. That is

$$R\hat{x} = -\hat{x}R\,, \qquad R\hat{p} = -\hat{p}R\,. \tag{2}$$

Let us note that throughout the paper we will use units where Planck's constant is set to unity, i.e. $\hbar = 1$. The algebra (1) is usually referred as the Wigner algebra.

In the coordinate representation the momentum operator is given by

$$\hat{p} = \frac{1}{\mathrm{i}}\partial_x^{Dunkl}, \tag{3}$$

with Dunkl derivative [7] defined as

$$\partial_x^{Dunkl} = \partial_x + \frac{\nu}{x}\,(1 - R)\ . \tag{4}$$

Here $\partial_x = \frac{\partial}{\partial x}$ denotes the usual derivative with respect to the coordinate x.

Quantum mechanics involving the Dunkl derivative was originally introduced in [8] by putting the stationary Schrödinger equation in the form

$$\left(-\frac{1}{2m}\left(\partial_x^{Dunkl}\right)^2 + V(x)\right)\psi(x) = E\psi(x) \tag{5}$$

It is well known [8] that the momentum operator (3) is not Hermitian when working in the usual Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R}, \mathrm{d}x)$ equipped with the standard Lebesgue measure $\mathrm{d}x$ on the real line. However, it is Hermitian on $\mathcal{H}_\nu = L^2(\mathbb{R}, \mathrm{d}\mu)$ equipped with a weighted measure $\mathrm{d}\mu(x) = |x|^{2\nu}\mathrm{d}x$. For some applications within $\mathcal{H}_\nu$ we refer to refs. [7–13].

The objective of the following section is to construct a Hermitian momentum operator on the standard Hilbert space $\mathcal{H}_0$. We will also present a modified continuity equation for WDQM.

III. HERMITIAN MOMENTUM IN DUNKL QUANTUM MECHANICS

From now on we will exclusively work in the Hilbert space $\mathcal{H}_0$ equipped with the usual Lebesgue measure $\mathrm{d}x$. That is, the inner product of two pure states (or wave functions) ϕ and ψ is given by

$$(\phi, \psi) = \int_{-\infty}^{\infty} \mathrm{d}x\, \phi^*(x)\psi(x)\,. \tag{6}$$

and the expectation value of an observable represented by a Hermitian operator $\hat{A}$ in a given quantum state ψ is defined as

$$\langle \hat{A} \rangle_\psi = \int_{-\infty}^{\infty} \mathrm{d}x\, \psi^*(x)\hat{A}\psi(x) \tag{7}$$

Obviously the momentum operator $\hat{p} = -\mathrm{i}\partial_x^{Dunkl}$ is not Hermitian due to the additional term $\frac{\nu}{\mathrm{i}x}$ showing up in (3). However, with the non-Hermitian $\hat{p}$ we can construct a Hermitian momentum operator via

$$\hat{P} = \frac{1}{2}\left(\hat{p} + \hat{p}^\dagger\right), \tag{8}$$

whose coordinate representation is given by (Note: On $\mathcal{H}_0$ the reflection operator is Hermitian as $(\phi, R\psi) = (R\phi, \psi)$).

$$\hat{P} = \frac{1}{\mathrm{i}} D_x = \frac{1}{\mathrm{i}} \left(\partial_x - \frac{\nu}{x} R \right) . \tag{9}$$

This operator was first introduced in [6]. From now on we refer to the anti-Hermitian operator D_x as the symmetric Dunkl derivative.

The Hermitian momentum operator (8) obeys the Wigner algebra

$$[\hat{x}, \hat{P}] = \mathrm{i}\,(1 + 2\nu R) . \tag{10}$$

We also note the anti-commutation relations

$$\{R, \hat{x}\} = 0 = \{R, \hat{P}\} . \tag{11}$$

Then, the time-dependent Hermitian Dunkl-Schrödinger equation for an external scalar potential V may be represented by

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t) = \left(\frac{\hat{P}^2}{2m} + V(x) \right) \psi(x,t) . \tag{12}$$

Or more explicitly

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{1}{2m} D_x^2 + V(x) \right] \psi(x,t) \tag{13}$$

$$= \left[-\frac{1}{2m}\partial_x^2 - \frac{1}{2m}\left(-\frac{\nu^2}{x^2} + \frac{\nu}{x^2} R \right) + V(x) \right] \psi(x,t) . \tag{14}$$

With the usual ansatz

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}Et}\psi(x) , \tag{15}$$

the stationary Hermitian Dunkl-Schrödinger equation with potential V reads

$$\hat{H}\psi(x) = \left(-\frac{1}{2m} D_x^2 + V(x) \right) \psi(x) = E\psi(x) . \tag{16}$$

In this paper we will exclusively consider symmetric potentials only, that is, we request

$$(RV)(x) = V(x) \qquad \Longleftrightarrow \qquad [R, V(\hat{x})] = 0 . \tag{17}$$

For such symmetric potentials the Hamiltonian commutes with the reflection operator. In other words, we can find simultaneous eigenfunction $\psi_{E,s}$ of the Hamiltonian $\hat{H}$ and the reflection operator R with

$$\hat{H}\psi_{E,s}(x) = E\psi_{E,s}(x) \qquad \text{and} \qquad R\psi_{E,s}(x) = s\psi_{E,s}(x) . \tag{18}$$

Note that $s \in \{-1, 1\}$ with $s = 1$ for even states and $s = -1$ for odd states. This implies that we can split our Hilbert space $\mathcal{H}_0$ into two subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$ belonging to even and odd states, respectively. In other words, R can act as a Z_2-grading operator on $\mathcal{H}_0 = \mathcal{H}^+ \oplus \mathcal{H}^-$ [18].

In concluding this section, let us establish a generalized continuity equation for solution of the Hermitian Dunkl-Schrödinger equation by utilizing the symmetric Dunkl derivative (9),

$$\frac{\partial \rho(x,t)}{\partial t} + D_x J(x,t) = 0 . \tag{19}$$

Here

$$\rho(x,t) = |\psi(x,t)|^2 \tag{20}$$

is the usual time-dependent probability density for an arbitrary solution of (12), which is normalized to unity

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \rho(x,t) = \int_{-\infty}^{\infty} \mathrm{d}x\, |\psi(x,t)|^2 = 1 \tag{21}$$

and thus obeys

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} \mathrm{d}x\, \rho(x,t) = 0\,. \tag{22}$$

However, for the time-dependent probability current we choose a generalized definition via the symmetric Dunkl derivative (9),

$$J(x,t) = \frac{1}{2m\mathrm{i}} \left(\psi^*(x,t) D_x \psi(x,t) - \psi(x,t) D_x \psi^*(x,t)\right)\,. \tag{23}$$

We leave it to the reader to verify that the continuity equation (19) is fulfilled as well as the Ehrenfest relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{x}\rangle_\psi = \frac{1}{m}\langle \hat{P}\rangle_\psi\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{P}\rangle_\psi = -\langle V'(x)\rangle_\psi\,. \tag{24}$$

Note that for symmetric V we have $[\hat{P},\hat{H}] = [\hat{P},V] = -\mathrm{i}[\partial_x, V] = -\mathrm{i}V'(x)$ as $[R,V]=0$.

IV. CONTINUITY CONDITIONS OF WAVE FUNCTIONS

In this section we will study the continuity conditions which must be obeyed by the wave function at points where the potential has a discontinuity.

To begin with, we first investigate the inversion of the symmetric Dunkl derivative. As is evident, D_x does not have the ordinary integral as its inverse operation. Thus, we propose for an arbitrary piecewise differentiable function F, the ansatz

$$\int^x \mathrm{d}y\, w(y) D_y F(y) = \int^x \mathrm{d}y\, w(y)\left(\partial_y - \frac{\nu}{y}R_y\right) F(y), \tag{25}$$

where we have introduced a symmetric weight function $w(x) = w(-x)$. It turns out that we need to consider the cases of odd and even F separately.

Case $(RF)(x) = F(x)$:

In this case the function F is even and integration by parts of (25) results in

$$\begin{aligned}\int^x \mathrm{d}y\, w(y) D_y F(y) &= \int^x \mathrm{d}y\, w(y)\left(\partial_y - \frac{\nu}{y}\right) F(y) \\ &= w(x)F(x) - \int^x \mathrm{d}y\left(w'(y) + \frac{\nu}{y}w(y)\right) F(y)\,.\end{aligned} \tag{26}$$

With the choice

$$w(x) = |x|^{-\nu}, \tag{27}$$

the last integral vanishes and we get

$$|x|^\nu \int^x \mathrm{d}y\, |y|^{-\nu} D_y F(y) = F(x)\,. \tag{28}$$

In other words, in the subspace $\mathcal{H}^+$ the inverse of the symmetric Dunkl derivative is given by the integral operator

$$D_x^{-1}(\cdot) = |x|^\nu \int^x \mathrm{d}y\, |y|^{-\nu}(\cdot) \tag{29}$$

Case $(RF)(x) = -F(x)$**:**
As in the previous case we do an integration by parts and arrive at

$$\begin{aligned}\int^x \mathrm{d}y\, w(y) D_y F(y) &= \int^x \mathrm{d}y\, w(y) \left(\partial_y + \frac{\nu}{y}\right) F(y) \\ &= w(x)F(x) - \int^x \mathrm{d}y \left(w'(y) - \frac{\nu}{y} w(y)\right) F(y)\,.\end{aligned} \tag{30}$$

Now we need to choose

$$w(x) = |x|^\nu \tag{31}$$

to make the last integral disappear and get

$$|x|^{-\nu} \int^x \mathrm{d}y\, |y|^\nu D_y F(y) = F(x)\,. \tag{32}$$

Hence, on the subspace $\mathcal{H}^-$ the inverse of the symmetric Dunkl derivative is given by the integral operator

$$D_x^{-1}(\cdot) = |x|^{-\nu} \int^x \mathrm{d}y\, |y|^\nu (\cdot) \tag{33}$$

We are now in a position to study the continuity conditions of the wave function at locations where the symmetric potential has discontinuities, say at $x_0 = \pm a$ with $a > 0$. The time-independent Hermitian Dunkl-Schrödinger equation reads

$$D_x^2 \psi_{E,s}(x) = 2m(V(x) - E)\psi_{E,s}(x), \tag{34}$$

where $V(x)$ is symmetric and has symmetric finite discontinuities at $x_0 = \pm a$, that is,

$$\lim_{\varepsilon\to 0} V(x_0 + \varepsilon) \neq \lim_{\varepsilon\to 0} V(x_0 - \varepsilon)\,, \tag{35}$$

with both limits being finite.
We will now apply the inverse operator D_x^{-1} on both sides of eq. (34). For doing so we need to discuss the two case $s = +1$ and $s = -1$ separately.
For an even wave function $\psi_{E,+1}$ its symmetric Dunkl derivative $D_x\psi_{E,+1}$ is an odd function and we need to use the integral operator (33). Applying this operator on both sides of (34) at $x = x_0 + \varepsilon$ and $x = x_0 - \varepsilon$ and taking the difference leads us to

$$\left[|x|^\nu \left(D_x \psi_{E,+1}\right)(x)\right]_{x=x_0-\varepsilon}^{x=x_0+\varepsilon} = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \mathrm{d}y\, |y|^\nu (V(y) - E)\psi_{E,+1}(y) \tag{36}$$

In the limit $\varepsilon \to 0$, the right-hand side of above equation vanishes. Observing that the weight function $w(x) = |x|^\nu$ is continuous at $x_0 \neq 0$, the continuity condition for the symmetric Dunkl derivative $D_x\psi_{E,+1}$ is given by

$$\lim_{\varepsilon\to 0}(D_x\psi_{E,+1})(x_0 + \varepsilon) = \lim_{\varepsilon\to 0}(D_x\psi_{E,+1})(x_0 - \varepsilon)\,. \tag{37}$$

For an odd wave function $\psi_{E,-1}$, we do the same steps as above, however using integral operator (29) which leads us to

$$\left[|x|^{-\nu} \left(D_x \psi_{E,-1}\right)(x)\right]_{x=x_0-\varepsilon}^{x=x_0+\varepsilon} = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \mathrm{d}y\, |y|^{-\nu} (V(y) - E)\psi_{E,-1}(y) \tag{38}$$

Again the weight function $w(x) = |x|^{-\nu}$ is continuous at $x_0 \neq 0$ and taking the limit $\varepsilon \to 0$ we arrive at the same continuity condition now applied to odd wave function.

$$\lim_{\varepsilon\to 0}(D_x\psi_{E,-1})(x_0 + \varepsilon) = \lim_{\varepsilon\to 0}(D_x\psi_{E,-1})(x_0 - \varepsilon)\,. \tag{39}$$

Let us now take a look at the probability current J. For a stationary solution the probability density ρ does not depend on time and the continuity equation (19) imposes the condition

$$(D_x J)(x) = 0\,. \tag{40}$$

Note that for even as well as odd solution ψ the probability current (23) will always be odd. Hence, we apply (29) to (40) at both positions, $x = x_0 + \varepsilon$ and $x = x_0 - \varepsilon$, and arrive at the continuity condition of the probability current

$$\lim_{\varepsilon\to 0} J(x_0+\varepsilon) = \lim_{\varepsilon\to 0} J(x_0-\varepsilon)\,. \tag{41}$$

Let us emphasize that this only holds for stationary states with a well defined parity. Together with the continuity condition for the symmetric Dunkl derivative, the above also implies continuity of the wave functions at x_0.

Thus, we have two continuity relations for the simultaneous eigenstates $\psi_{E,s}$ of parity operator R and Hamiltonian $\hat{H}$ with symmetric potential possessing a finite discontinuity at x_0:

$$\lim_{\varepsilon\to 0} \psi_{E,s}(x_0-\varepsilon) = \lim_{\varepsilon\to 0} \psi_{E,s}(x_0+\varepsilon)\,, \tag{42}$$

$$\lim_{\varepsilon\to 0} (D_x\psi_{E,s})(x_0-\varepsilon) = \lim_{\varepsilon\to 0} (D_x\psi_{E,s})(x_0+\varepsilon)\,. \tag{43}$$

In concluding this section, let us briefly look at the behavior of the wave function near the origin, that is at $|x| \ll 1$. For a well-behaved finite potential V near the origin, the ansatz $\psi \sim x^\alpha$ results in $\alpha = \nu$ for the even solutions and $\alpha = \nu + 1$ for odd solutions. That is, for $|x| \ll 1$ we have

$$\psi_{E,+1}(x) \sim |x|^\nu\,, \qquad \text{and} \qquad \psi_{E,-1}(x) \sim x\,|x|^\nu\,, \tag{44}$$

which are both square integrable at $x = 0$ if $\nu > -1/2$.

V. THE FINITE POTENTIAL WELL

As an explicit example, let us consider a finite potential well with depths $-V_0$ in a finite range $-a < x < a$. That is, we consider the symmetric potential

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| > a \end{cases}\,, \tag{45}$$

where both parameters, V_0 and a are positive real numbers. Since $V(x)$ is symmetric, it is sufficient to consider the region $x > 0$ only. Here we will limit our discussion to bounded solution with $-V_0 < E < 0$.

First, let us look at solutions of the stationary Dunkl-Schrödinger equation in the range $0 < x < a$,

$$(D_x^2 + k^2)\psi_{E,s}(x) = 0, \tag{46}$$

Here we have set

$$k = \sqrt{2m(V_0+E)} > 0 \tag{47}$$

and eq. (46) can be written as

$$\partial_x^2\psi_{E,s} + \left(\frac{\nu(s-\nu)}{x^2} + k^2\right)\psi_{E,s} = 0 \tag{48}$$

where we have used $R\psi_{E,s}(x) = s\psi_{E,s}(x)$. With the help of the replacement $\psi_{E,s}(x) = \sqrt{x}\,u(x)$, the above equation is turned into Bessel's differential equation

$$x^2u'' + xu' + \left(k^2x^2 - \left(\nu - \frac{s}{2}\right)^2\right)u = 0, \tag{49}$$

whose linearly independent solutions are given by Bessel functions of first and second kind, denoted by $J_{\nu-\frac{s}{2}}(kx)$ and $Y_{\nu-\frac{s}{2}}(kx)$, respectively. The latter one we have to discard as it does not fulfil the boundary condition (44). In conclusion, the even ($s = 1$) and odd ($s = -1$) solutions in the range $x \in [-a, a]$ are given by

$$\psi_{E,+1}(x) = A_+\sqrt{|x|}J_{\nu-\frac{1}{2}}(k|x|)\,, \qquad \psi_{E,-1}(x) = A_-\frac{x}{\sqrt{|x|}}J_{\nu+\frac{1}{2}}(k|x|)\,, \tag{50}$$

which obey conditions (44). The positive real constants $A_\pm$ are determined at a later stage via the normalization and continuity conditions (42) and (43).

Now let us consider the case where $x > a$. In this case the stationary Dunkl-Schrödinger equation reads

$$(D_x^2 - \kappa^2)\psi_{E,s}(x) = 0, \tag{51}$$

where we set

$$\kappa = \sqrt{2m|E|} > 0\,. \tag{52}$$

The same substitution as in the previous case now results in the modified Bessel equation with the two linearly independent solutions given by the modified Bessel functions of first and second kind, denoted by $I_{\nu-\frac{s}{2}}(\kappa x)$ and $K_{\nu-\frac{s}{2}}(\kappa x)$, respectively. They exhibit for large argument the following asymptotic behavior

$$I_\alpha(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + \mathcal{O}\left(z^{-1}\right)\right)\,, \qquad K_\alpha(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + \mathcal{O}\left(z^{-1}\right)\right)\,, \tag{53}$$

and hence only the solution of the second kind is admissible. Therefore, we are led to the even and odd solution in the range $|x| > a$

$$\psi_{E,+1}(x) = B_+\sqrt{|x|}K_{\nu-\frac{1}{2}}(\kappa|x|)\,, \qquad \psi_{E,-1}(x) = B_-\frac{x}{\sqrt{|x|}}K_{\nu+\frac{1}{2}}(\kappa|x|)\,, \tag{54}$$

where as above the two positive real constants $B_\pm$ are to be determined in the following.

With the explicit eigenfunctions at hand we can now determine the corresponding eigenvalues by utilizing the two continuity conditions (42) and (43) at $x_0 = a$. We will discuss the even and odd case separately.

A. Eigenvalues for even solutions

For the even wave function the continuity conditions result in the two equation

$$A_+J_{\nu-\frac{1}{2}}(ka) = B_+K_{\nu-\frac{1}{2}}(\kappa a) \qquad \text{and} \qquad A_+kJ_{\nu+\frac{1}{2}}(ka) = B_+\kappa K_{\nu+\frac{1}{2}}(\kappa a)\,. \tag{55}$$

Eliminating the two constants A_+ and B_+ we arrive at

$$k\left(\frac{J_{\nu+\frac{1}{2}}(ka)}{J_{\nu-\frac{1}{2}}(ka)}\right) = \kappa\left(\frac{K_{\nu+\frac{1}{2}}(\kappa a)}{K_{\nu-\frac{1}{2}}(\kappa a)}\right)\,. \tag{56}$$

Together with $\kappa = \sqrt{2mV_0 - k^2}$ these relations determine admissible values for k. Hence, the solutions denoted by k_n, $n = 1, 2, \ldots, n_0$ provide us with the eigenvalues $E_{n,+1} = -V_0 + \frac{k_n^2}{2m}$. Note that n_0 ist the maximal n where $E_{n,+1}$ remains negative. As above equation (56) cannot be solved analytically we provide here a graphical solution by introducing dimensionless variables $\xi = ka > 0$, $b = a\sqrt{2mV_0} > 0$ and real functions

$$f_+(\xi) = \xi\left(\frac{J_{\nu+\frac{1}{2}}(\xi)}{J_{\nu-\frac{1}{2}}(\xi)}\right)\,, \qquad g_+(\xi) = \sqrt{b^2 - \xi^2}\left(\frac{K_{\nu+\frac{1}{2}}\left(\sqrt{b^2-\xi^2}\right)}{K_{\nu-\frac{1}{2}}\left(\sqrt{b^2-\xi^2}\right)}\right)\,. \tag{57}$$

The energy eigenvalues corresponding to the even solutions are then determined by the solutions $\xi_n = ak_n$ of $f_+(\xi) = g_+(\xi)$.

The function $f_+(\xi)$ is independent of parameter b and has positive zeros of order one at $\alpha_{\nu+\frac{1}{2},p}$ with $p = 1, 2, 3, \ldots$, where $\alpha_{\mu,p}$ denotes the p-th zero of the Bessel function $J_\mu(\alpha_{\mu,p}) = 0$. For $\xi \ll 1$ it behaves like $f_+(\xi) \sim \xi^2/(2\nu + 1)$. In addition, $f_+(\xi)$ has simple poles at $\alpha_{\nu-\frac{1}{2},p}$ with $p = 1, 2, 3, \ldots$. The function f_+ is shown in figure 1 in purple color. Let us also note that the zeros of the two Bessel functions used in definition of f_+ are interlaced as follows

$$\alpha_{\nu-\frac{1}{2},1} < \alpha_{\nu+\frac{1}{2},1} < \alpha_{\nu-\frac{1}{2},2} < \alpha_{\nu+\frac{1}{2},2} < \alpha_{\nu-\frac{1}{2},3} < \cdots\,. \tag{58}$$

This implies that between two zeros of f_+ we always have a simple pole.

Now let us take a look at the function $g_+(\xi)$. Its behaviour is dominated by the factor $\sqrt{b^2-\xi^2}$ from which we conclude that $\xi \leq b$. At $\xi = 0$ it starts with a positive value given by $g_+(0) = bK_{\nu+\frac{1}{2}}(b)/K_{\nu-\frac{1}{2}}(b) > f_+(0) = 0$. For $-1/2 < \nu < 1/2$ it vanishes at $\xi = b$. More precisely, it behaves like[1] $g_+(b-\epsilon) \sim \frac{\Gamma(1/2+\nu)}{\Gamma(1/2-\nu)} 2^{\nu+1/2} b^{1/2-\nu} \epsilon^{1/2-\nu}$. For $\nu > 1/2$ the function g_+ terminates at $\xi = b$ with a positive value $g_+(b) = 2\nu - 1$. In figure 1 we have plotted g_+ for $\nu = 0.2$ and $\nu = 0.7$ for various values of parameter b. The number of even bound states increases as V_0 increases but decreases as ν increases. Indeed, for the range $|\nu| < 0.5$

$$\alpha_{\nu+\frac{1}{2},p} < \sqrt{2mV_0a^2} < \alpha_{\nu+\frac{1}{2},p+1}, \tag{59}$$

we have $n_0 = p + 1$ even bound states, where we set $\alpha_{\nu+\frac{1}{2},0} = 0$. In other words for $|\nu| < 0.5$ at least one even bound state exists. However, if $\nu > 1/2$ the number of bound states might be one less as the function g_+ terminates at $\xi = b$ with positive value $g_+(b) = 2\nu - 1 > 0$. Hence, if $f_+(b) < 2\nu - 1$ the last eigenvalue is not present. Hence, for b sufficiently small there may not exist any even bound state all.

B. Eigenvalues for odd solutions

In the case of odd eigenfunctions the continuity conditions at $x = a$ result in

$$A_- J_{\nu+\frac{1}{2}}(ka) = B_- K_{\nu+\frac{1}{2}}(qa)\,, \qquad kA_- J_{\nu-\frac{1}{2}}(ka) = -\kappa B_- K_{\nu-\frac{1}{2}}(qa)\,, \tag{60}$$

which gives us the relation

$$k\left(\frac{J_{\nu-\frac{1}{2}}(ka)}{J_{\nu+\frac{1}{2}}(ka)}\right) = -\kappa\left(\frac{K_{\nu-\frac{1}{2}}(\kappa a)}{K_{\nu+\frac{1}{2}}(\kappa a)}\right). \tag{61}$$

As before we introduce two functions f_- and g_- given by

$$f_-(\xi) = -\xi\left(\frac{J_{\nu-\frac{1}{2}}(\xi)}{J_{\nu+\frac{1}{2}}(\xi)}\right), \qquad g_-(\xi) = \sqrt{b^2-\xi^2}\left(\frac{K_{\nu-\frac{1}{2}}\left(\sqrt{b^2-\xi^2}\right)}{K_{\nu+\frac{1}{2}}\left(\sqrt{b^2-\xi^2}\right)}\right), \tag{62}$$

and the eigenvalues of the odd bound states are determined from the intersection of these two functions.

Noting that f_- is related to f_+ via $f_-(\xi) = -\xi^2/f_+(\xi)$, we observe that the positive zeros of f_- are now located at the poles of f_+ and the poles of f_- are now at the positions of the positive zeros of f_+. We also observe that f_- starts with a negative value, that is, $f_-(0) = -(2\nu + 1)$.

The function g_- is also related to g_+ via $g_-(\xi) = \left(b^2-\xi^2\right)/g_+(\xi)$. As above this leads to the bound $\xi \leq b$. At the origin it starts out with the positive value $g_-(0) = bK_{\nu-\frac{1}{2}}(b)/K_{\nu+\frac{1}{2}}(b) > 0$ and vanishes at $\xi = b$. To be more explicit, for $-1/2 < \nu < 1/2$ it behaves like $g_+(b-\epsilon) \sim \frac{\Gamma(1/2-\nu)}{\Gamma(1/2+\nu)} 2^{-\nu+1/2} b^{1/2+\nu} \epsilon^{1/2+\nu}$, whereas for $\nu > 1/2$ the behavior is of the form $g_-(b-\epsilon) \sim \frac{2b\epsilon}{2\nu-1}$. Hence for $b < \alpha_{\nu-\frac{1}{2},1}$ we even have no odd bound state. In the general case the number of odd bound states increases as V_0 increases. Indeed, for

$$\alpha_{\nu-\frac{1}{2},p} < \sqrt{2mV_0a^2} < \alpha_{\nu-\frac{1}{2},p+1}\,. \tag{63}$$

we have p odd bound states. In figure 2 we have plotted g_- for $\nu = 0.2$ and same values of parameter b as in figure 1.

VI. CONCLUSION

The first objective of the present work was the construction of a Hermitian momentum operator of WDQM in the standard Hilbert space with the usual measure. This was achieved in section 3 by symmetrizing the

[1] Note that for small $z > 0$ and $\mu > 0$ the modified Bessel function behaves like $K_\mu(z) \sim \frac{\Gamma(\mu)}{2}\left(\frac{2}{z}\right)^\mu$. For $\mu < 0$ we may use the relation $K_\mu(z) = K_{-\mu}(z)$.

non-Hermitian momentum operator and expressing it in terms of the symmetric Dunkl derivative D_x. In section 4 we first constructed the inverse operator D_x^{-1} and found that it has different realizations in the subspace of even and odd functions. As any function can be decomposed into its even and odd part, this operator is trivially extended to the full Hilbert space. The symmetric Dunkl derivative also led us to a modified continuity equation and Ehrefest's theorem in WDQM.

The inverse operator D_x^{-1} then enabled us to study the continuity conditions of wave functions within WDQM in the presence of a discontinuous symmetric potential. These were presented in eqs. (42) and (43). As an explicit example the finite potential well was studied. As in ordinary quantum mechanics the eigenfunctions can be found in closed form but the associated eigenvalues may only be obtained in a numerical or graphical way. It was found that for $-\frac{1}{2} < \nu < \frac{1}{2}$ there always exists an even ground state. However, for $\nu > \frac{1}{2}$ and V_0 sufficiently small this potential well may not posses any bound state. The number of even and odd bound states increases with increasing V_0 but decreases with increasing deformation parameter ν as the positive zeros of the Bessel functions $J_{\nu\pm\frac{1}{2}}(\xi)$ move to the right on the ξ-axis with increasing index ν.

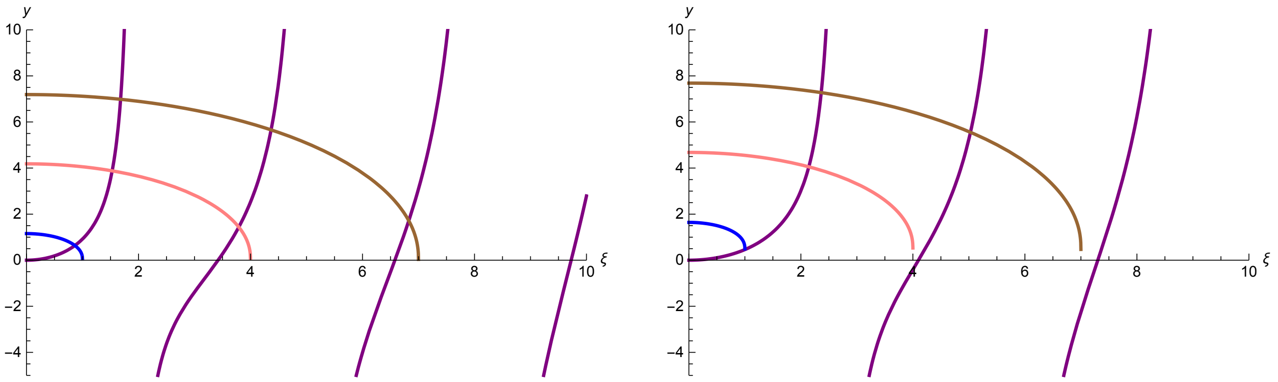

FIG. 1: Plot of function f_+ and g_+ as defined in (57) for $\nu = 0.2$ on the left and for $\nu = 0.7$ on the right. The function f_+ is shown in Purple and $g_+(\xi)$ is shown for $b = 1$ in Blue, $b = 4$ in Pink and for $b = 7$ in Brown.

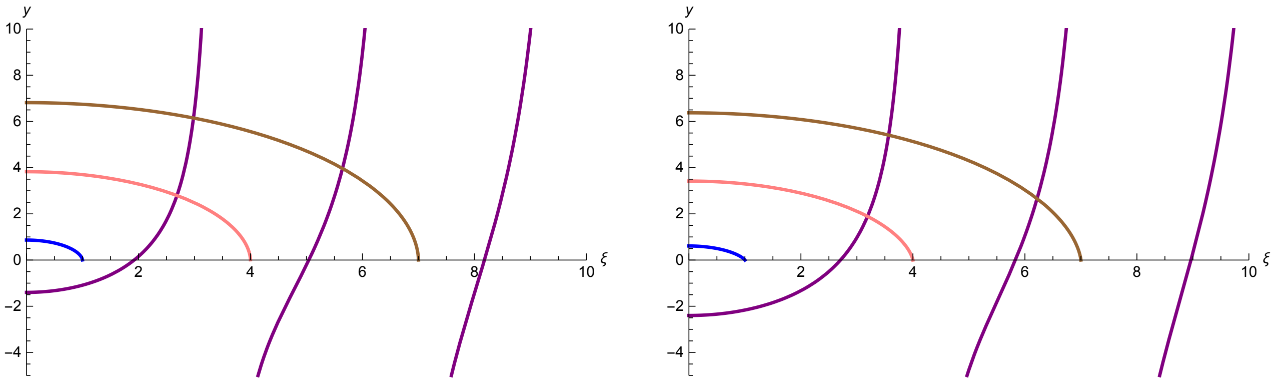

FIG. 2: Plot of function f_- and g_- as defined in (62) for $\nu = 0.2$ on the left and for $\nu = 0.7$ on the right. As in figure 1, function f_+ is shown in Purple and $g_+(\xi)$ is shown for $b = 1$ (Blue), $b = 4$ (Pink) and $b = 7$ (Brown).

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